# Investigation of a Lagrangian Description of a Modified Demonstration of Atwood's Machine

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Abstract

Despite the surprising accuracy of the predictions made by Newtonian Mechanics, its calculations necessitate keeping track of a multitude of variables, each varying according to the Laws of Motion. Lagrangian mechanics aim to reformulate such calculations using physical constraints; resulting in a simpler description of motion within a coordinate system, which can be obtained from the solution of differential equations. In this paper, I aim to describe the motion of a modified Atwood's machine with an additional angle of freedom of the smaller mass—a problem referred to as the "Matchbox and the Keychain"—using Lagrangian mechanics, and empirically comparing the two crucial variables in its motion: an *angle* and a *length,* with calculated theoretical values.

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# I. Background

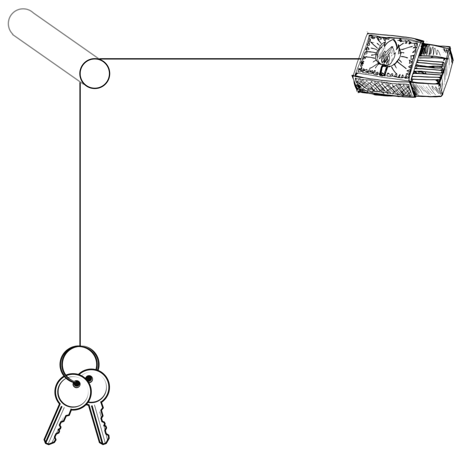
## 1. Lagrangian Mechanics

Lagrangian Mechanics, simply put, are a reformulation of Newtonian Mechanics, developed with the intent of reducing complexity in its calculations. It includes a description of a system with equations describing its constraints using, for example, the Principle of Conservation of Energy. Equations describing such constraints can be mathematically manipulated into Euler-Lagrange equations, whose solutions reveal relationships between variables, describing the evolution of the mechanical system.

## 2. The Matchbox and the Keychain

The problem of the Matchbox and the Keychain was introduced in the American Journal of Physics in 1991, with the title: "A Surprising Physics Demonstration." This problem includes two bodies whose masses are different orders of magnitude, connected by a string hung on a fixed pole held horizontally—similar to an Atwood's machine with a pulley, but where the fixed axis does not rotate, and the smaller mass does not hang towards the direction of gravity. The distance between the smaller mass and the pole is denoted $r$, and the angle between the reference axis $x$ and the string between the smaller mass and the pole, is denoted $\theta$.

*[Figure 1.2.1. The Matchbox and the Keychain Demonstration]*



Holding the smaller mass so that it is nearly the height of the pole, and letting go, would cause the bigger mass to initially fall; only shortly later to the stopped by the friction of the smaller mass wrapping around the pole multiple times. This is a rather counter-intuitive behavior, but one that nevertheless obeys the laws of mechanics, and therefore, whose evolution is calculable and predictable.

Describing the motion of the bodies in the aforementioned demonstration is most elegantly done through Lagrangian Mechanics, explored extensively in Section II. An experiment of the demonstration, described in Section III, will be conducted and its results compared with the calculated predicted values. We can, therefore, formulate our research question thusly:

*In the physical demonstration of an Atwood machine with an additional angle of freedom between the smaller mass and the reference pole, how does the distance between the smaller mass and the pole vary according to the angle between the smaller mass and the pole?*

**3. Justification**

My personal interest in this topic stems from my surprise in the accuracy of Newtonian mechanics in describing the (classical) world—a minimal set of laws and assumptions, that predict the evolution of a system, theoretically, with great accuracy. I also find the Lagrangian description of this experiment fascinating, where such complex behavior can be predicted simply by using the law of conservation of energy. With the mathematical and computational tools and skills acquired in my other courses, I hope to combine these skills into this exploration to investigate this seemingly counter-intuitive behavior of the problem of the “Matchbox and the Keychain,” in order to gain an appreciation and understanding of the extent of the reaches of classical mechanics in describing the motions of bodies.

## II. Theoretical Outline

### 1. A Coordinate System

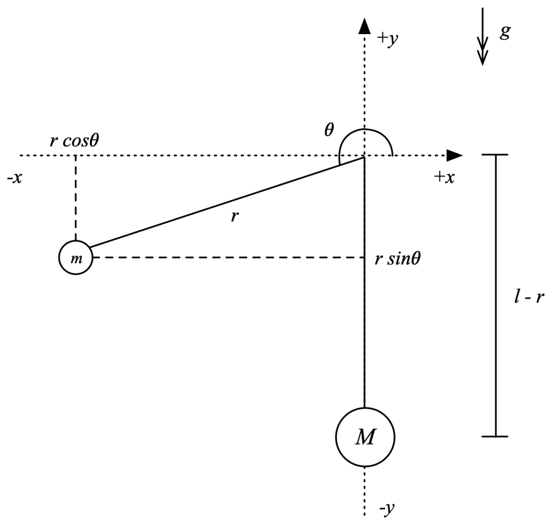
The demonstration concerns the following variables and constants:

$M$ : mass of the more massive body
$m$ : mass of the less massive body
$g$ : gravitational field strength
$l$ : length of the entire string
$r$ : distance between the smaller mass and the pole
$\theta$ : angle between the reference axis $x$ and the string between the smaller mass and the pole

We assume that:
- the thickness of the pole is zero (accounted for later in uncertainties)
- there is no friction in the system
- the masses are point masses

The two masses are plotted against a Cartesian coordinate system where the pole lies in the origin, while the reference axis is the positive *x-axis*. The direction of gravitational acceleration is towards the negative $y$ direction. *Diagram 2.1.1* is a lateral view of this hypothetical system.

*[Diagram 2.1.1: A Lateral View of the Matchbox and the Keychain Demonstration]*




The values of r and $\theta$ are alternate notations (polar coordinates) of the position of the masses. Trigonometrically, we can identify that $x = r \cdot \mathbf{cos}\theta$ and $y = r \cdot \mathbf{sin}\theta$.

### 2. Energy Constraints

As the sum of kinetic and potential energy in the system is constant, we can construct Lagrangian Equations that take advantage of this fact. Let:

$Y$ : the *y-axis* coordinate of mass M
$E_P$ : potential energy of the smaller and bigger mass
$E_K$ : kinetic energy of the smaller and bigger mass

Noting that $Y$ is always a negative value, it is equivalent to the negative difference between $l$ and $r$:

$$Y = -(l - r) = r - l$$

Each in their origin position, we can determine its potential energy from the gravitational potential energy equation $E_P = mgh$ (where $m$ is mass, $g$ is gravitational field strength, and $h$ is height relative to the reference axis). Note that values of $y$ and $Y$ can be negative, leading to a negative value of $E_P$.

$$E_P = mgy + MgY$$

$$= g(my + MY)$$

Kinetic energy can be derived from the equation $E_K = \frac{1}{2}mv^2$ (where $m$ is mass, and $v$ is the linear velocity of the mass). We use the time differential of position to denote linear velocity. The rate of change of $r$, equivalent to $\sqrt{x^2 + y^2}$ (obtainable by the Pythagorean theorem) is the linear velocity of mass $m$, while the rate of change of $Y$ is the linear velocity of mass $M$. We can therefore determine the sum of their kinetic energies to be:

*(Time differentials of values are denoted with dot notation)*

$$E_K = \frac{1}{2}m(\dot{\sqrt{x^2 + y^2}})^2 + \frac{1}{2}M\dot{Y}^2$$
$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{M}{2}\dot{Y}^2$$

Using identities $x = r \cdot \mathbf{cos}\theta$ and $y = r \cdot \mathbf{sin}\theta$, and $Y = r - l$ we can substitute $x$ and $y$ accordingly.

$$E_P = g(my + MY)$$
$$= g(mr\mathbf{sin}\theta + M(r - l))$$
$$= g(mr\mathbf{sin}\theta + Mr - Ml)$$

As $gMl$ is a constant term, and we are simply looking to constrain the value of $E_P$, we can ignore this term. Let us denote this new constant value as $E_{PC}$:

$$E_{PC} = gr(m\mathbf{sin}\theta + M)$$

We do the same for $E_K$. Note that $\dot{Y} = \dot{r}$, as $l$ is a constant. As the derivatives of *x, y,* and $Y$ are as follows:

$$\dot{x} = \frac{d}{dt}[r\,\mathbf{cos}\theta] = \dot{r}\,\mathbf{cos}\theta - r\,\mathbf{sin}\theta \cdot \dot{\theta}$$
$$\dot{y} = \frac{d}{dt}[r\,\mathbf{sin}\theta] = \dot{r}\,\mathbf{sin}\theta + r\,\mathbf{cos}\theta \cdot \dot{\theta}$$
$$\dot{Y} = \dot{r}$$

We can substitute from:

$$\begin{aligned} E_K &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{M}{2}\dot{Y}^2 \\ &= \frac{m}{2}(\dot{r}^2\mathbf{cos}^2\theta - 2r\dot{r}\,\mathbf{sin}\theta \cdot \dot{\theta} \cdot \mathbf{cos}\theta \cdot \dot{\theta} + r^2\mathbf{sin}^2\theta \cdot \dot{\theta}^2 + \\ &\qquad \dot{r}^2\mathbf{sin}^2\theta + 2r\dot{r}\,\mathbf{cos}\theta \cdot \dot{\theta} \cdot \mathbf{sin}\theta \cdot \dot{\theta} + r^2\mathbf{cos}^2\theta \cdot \dot{\theta}^2) + \frac{M}{2}\dot{r}^2 \\ &= \frac{m}{2}(\dot{r}^2(\mathbf{sin}^2\theta + \mathbf{cos}^2\theta) + r^2\dot{\theta}^2(\mathbf{sin}^2\theta + \mathbf{cos}^2\theta)) + \frac{M}{2}\dot{r}^2 \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{M}{2}\dot{r}^2 \\ &= \frac{1}{2}((m+M)\dot{r}^2 + mr^2\dot{\theta}^2) \end{aligned}$$

As we noted that $E_{PC}$ can be negative, we will sum $E_K$ and $-E_{PC}$ to find the Lagrangian constraint equation $L$:

$$\begin{aligned} L &= E_K - E_{PC} \\ &= \frac{1}{2}((m+M)\dot{r}^2 + mr^2\dot{\theta}^2) - gr(m\,\mathbf{sin}\theta + M) \end{aligned}$$

**3. Euler-Lagrange Equations**

Using this Lagrangian to determine the system's Euler-Lagrangian equations are an additionally laborious process including mathematical concepts I have yet not learned. Therefore, we will use Wolfram Mathematica, a mathematical computing software, to simply this task. The code used in Mathematica can be found in *Appendix A*.

< Code >
$\mathbf{lagr} = 0.5(m+M)r'(t)^2 + 0.5mr(t)^2\theta'(t)^2 - gr(t)(m\sin(\theta(t)) + M)$
$eq1 = \mathbf{EulerEquations}(\mathbf{lagr}, \theta(t), t)$
$eq2 = \mathbf{EulerEquations}(\mathbf{lagr}, r(t), t)$

The resulting two Euler-Lagrange Equations calculated by the program are:

$$mr\left(-g\cos\theta - 2\dot{r}\dot{\theta} - r\ddot{\theta}\right) = 0$$
$$-g(m\sin(\theta) + M) - (m+M)\ddot{r} + mr\dot{\theta}^2 = 0$$

Simplified:

$$-gr\cos\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$
$$(M+m)\ddot{r} = mr\dot{\theta}^2 - g(M + m\sin\theta)$$

These two differential equations can be solved with regards to any of the three variables, *r*, *θ*, or *t,* given initial conditions, to reveal the theoretical relationship between these values.

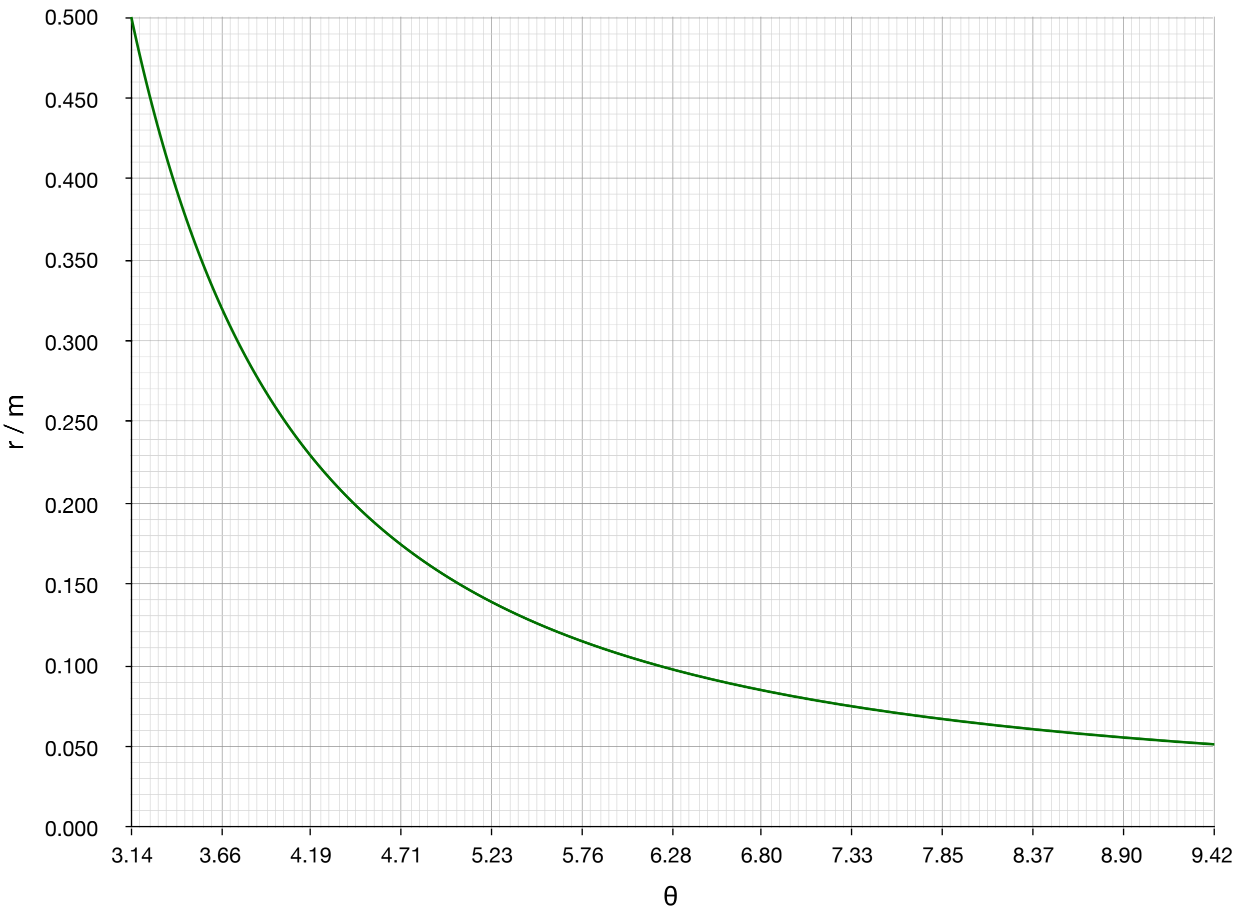
**4. Example Solutions**

The same program can be used to find solutions for these differential equations, given the values of initial conditions and constants. A numerical solution-finding feature for differential equations can be used to give points of very small intervals. For instance, if we were to set:

$M = 1kg$
$m = 0.01kg$
$g = 9.81$
**initial** $r = 0.5m$
**initial** $\theta = \pi$

The program outputs the following graph, of *r* plotted against *θ*, with time $0s < t < 0.33s$:

*[Graph 2.4.2: Theoretical Relationship of r and θ]*



The value of r is inversely correlated with $\theta$, which is to be expected, as the bigger mass will pull the smaller mass as the smaller mass moves in a roughly spiraling motion. Note, however, that this graph does not include compensation for friction. The decrease in r is purely due to energy changes within the system—converting the potential energy of the two masses to their rotational and linear kinetic energies.

We can now state our theoretical prediction as:

*Values of r will decrease in a rough exponential decay-like curve*[1] *against* $\theta$*, beginning at* $\theta = \pi$ *and* $r = 0.5m$*, and to* $\theta = 3\pi$ *where* $r \approx 0.05m$ *at* $t \approx 0.3s$*.*

Or more qualitatively:

*The mass will fall as the string wraps itself around the pole, returning to its original position (having wrapped itself one and a quarter times in relation to the bigger mass) after about 0.3 seconds. The length of the string between the pole and the smaller mass will decrease fast initially, and slow to a value of about 0.07m in this timespan.*

We will compare the data obtained from the experiment with this theoretical graph, quantitatively analyzing the fit of the data to the model using a calculation of the coefficient of determination. In our case, given the theoretical function $r_t(\theta_t)$, and the experimentally obtained values of $\theta$ and $r$ stored as a column vector, the coefficient is calculated as the following, where subscript $i$ denotes the *i-th* value in the vector, and $n$ is the number of data points:

$$R_{RES} = \Sigma_i[r_t(\theta_i) - r_i]$$ : Residual Sum of Squares

$$R_{TOT} = \Sigma_i[r_i - \frac{\Sigma_i r_i}{n}]$$ : Total Sum of Squares

$$R^2 = 1 - \frac{R_{RES}}{R_{TOT}}$$ : Coefficient of Determination

[1] Note that it is *not actually an exponential decay curve*. As the solution to the differential equations in the previous section was not found algebraically, this curve is a production of brute-force calculation, and therefore is impossible to identify currently.

## III. Experimental Setup

### 1. Protocol

A trial with masses $M$ and $m$ will be conducted, with the measurements of their respective values as well as the mass of the string in *Table 3.3.1*. The apparatus will be set up according to *Diagram 3.2.1*.

The string will be hung on a pole, held by stands fixed onto two tables using clamps, with enough force to reduce the movement of the pole itself due to the masses are constrained to under 0.01 meters. The metal pole is wrapped with smooth duct tape and liquid lubricant is applied to reduce friction at the contact point of the string.

A light beam from a projector at minimum 5 meters away and aligned with the pole *(Constraint A)* will cast a shadow of the moving smaller mass onto a graph paper whose graticule is 0.005 meters, and set up so that the perpendicular distance between the mass and the graphing paper is under 0.05 meters *(Constraint B)*. This shadow will be videotaped by a camera capable of recording at 60 frames per second, with the shutter speed less than 0.001 seconds.

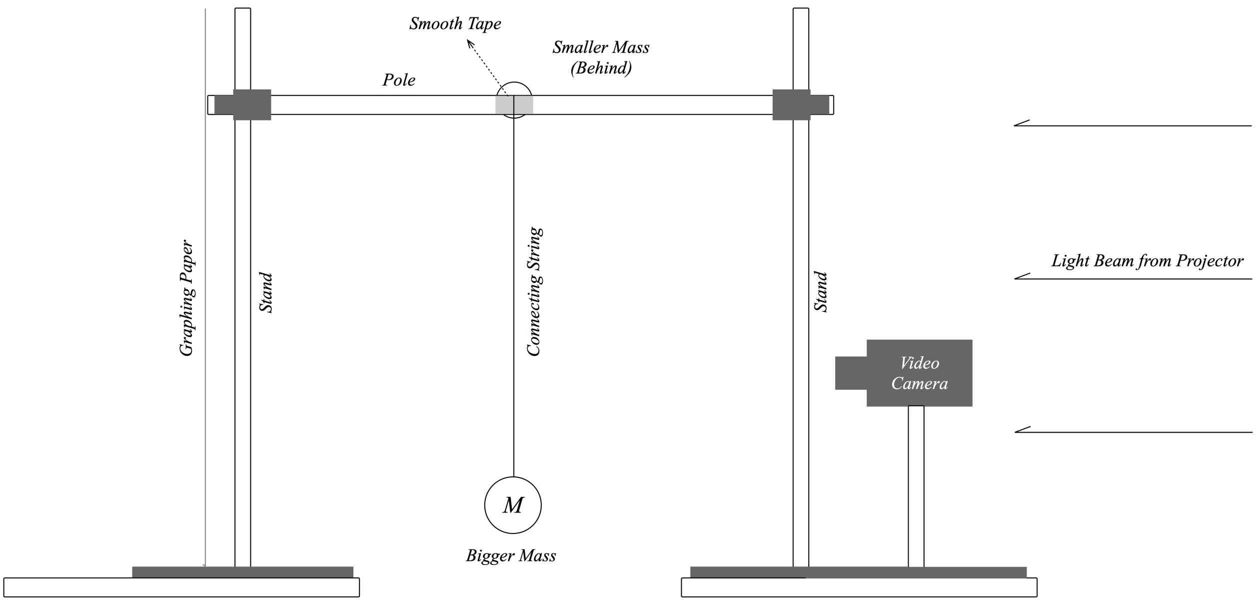
Each trial was performed according to the following simplified procedure:

1. Turn on the projector light
2. Position the string on the pole to satisfy *Constraint B*
3. Hold the smaller mass by its string, until the bigger mass is visibly at rest
4. Release the smaller mass as instantaneously as possible

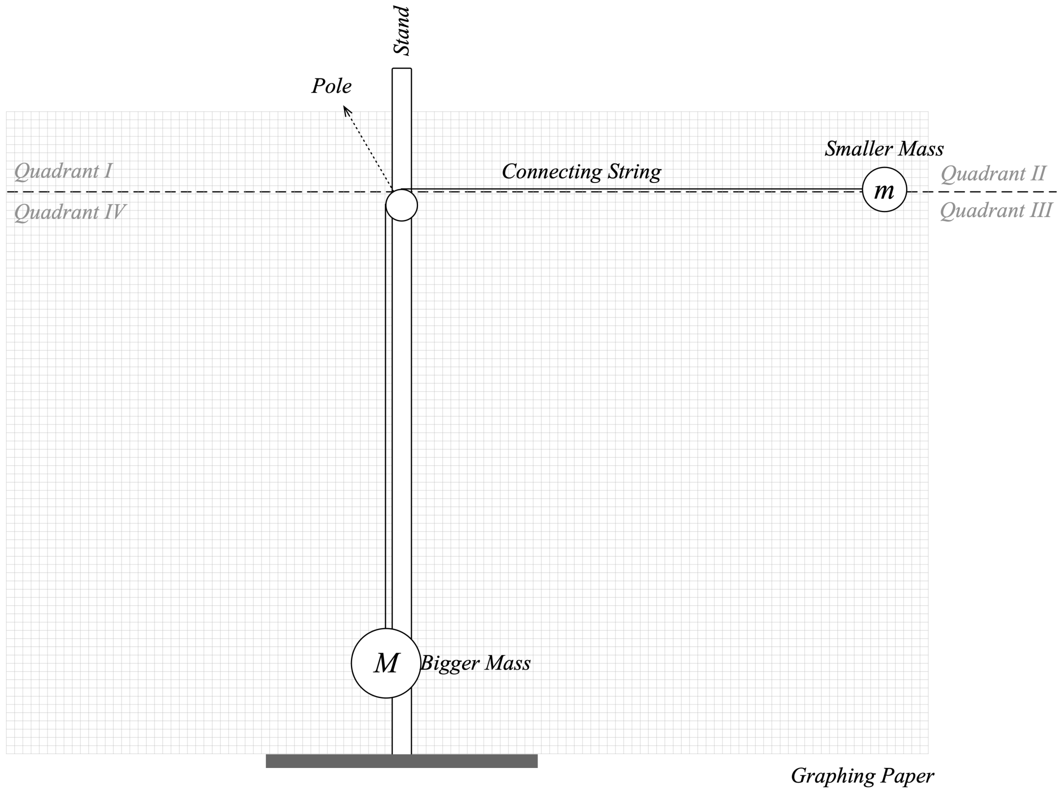
The uncertainty due to the light from the projector not being perpendicular to the graphing paper will be accounted for in *Section IV.*

### 2. Apparatus Setup

*[Diagram 3.2.1. Front View of Apparatus]*

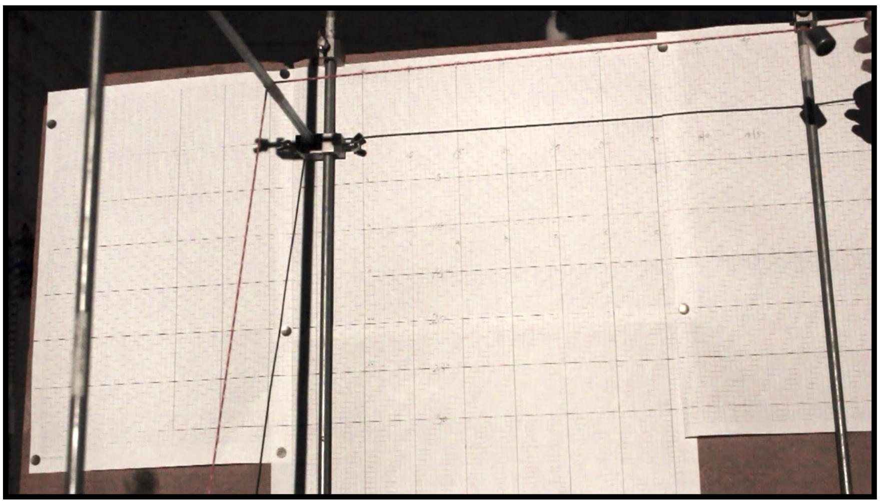


*[Diagram 3.2.2. Lateral View of Apparatus From the Projector]*




The view of the video camera filming the graph paper is shown in Figure 3.2.3. Note that the angle of the view of the camera does not matter, as only the shadow projected onto the graph paper is being measured, negating any parallax errors. Note also that the quadrants of the axes are unconventionally reflected along the *y*-axis to ease later calculations.

*[Figure 3.2.3. Example View from Camera of the Apparatus]*



### 3. Measured Variables

The masses were chosen to be 1.00 and 0.010 kilograms, with, respectively, ±1% and ±20% uncertainties.

*[Table 3.3.1. Measured Values of Controlled Variables and Constants]*

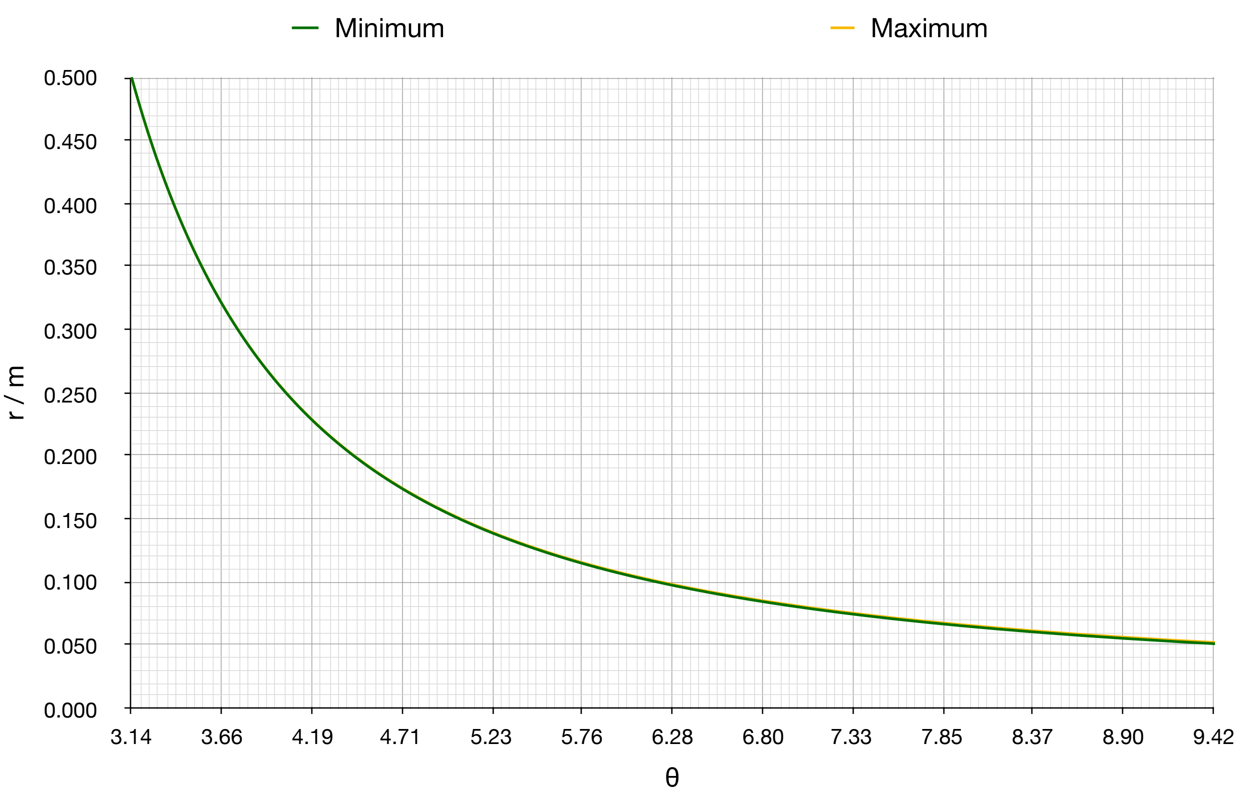
| Gravitational Field Strength g / $m \cdot s^{-2}$ | Mass of Larger Mass M / $kg$ | Mass of Smaller Mass m / $kg$ | Mass of String / kg |
|---|---|---|---|
| | ±1% | ±20% | ±0.0005 |
| 9.81 | 1.00 | 0.010 | 0.0020 |

As the uncertainty of the smaller mass encapsulates the full mass of the string, we need not take into account the mass of the string (however, this may lead to some problems. See *Evaluation*). With these uncertainties, the theoretical graph was calculated and plotted again, with all minimum and maximum combinations of $M$ and $m$, and the highest and lowest values of $r$ and $\theta$ were plotted:

| | | | |
|---|---|---|---|
| $M = 1.01kg$ | $M = 1.01kg$ | $M = 0.99kg$ | $M = 0.99kg$ |
| $m = 0.012kg$ | $m = 0.008kg$ | $m = 0.012kg$ | $m = 0.008kg$ |

This graph will be plotted with the experimental values in order to assess the correlation. Note, however, that the minimum and maximum values are indistinguishably close:

*[Graph 3.3.1. Theoretical Relationship of r and θ Taking Into Account Mass Uncertainties]*



## IV. RESULTS

### 1. Uncertainty Calculations

*A. Uncertainty Values of time t*

As the camera is capable of recording at 60 frames per second, and the exposure duration of one frame is at maximum 0.001 seconds, we can calculate the time coordinate of each captured frame by simply:

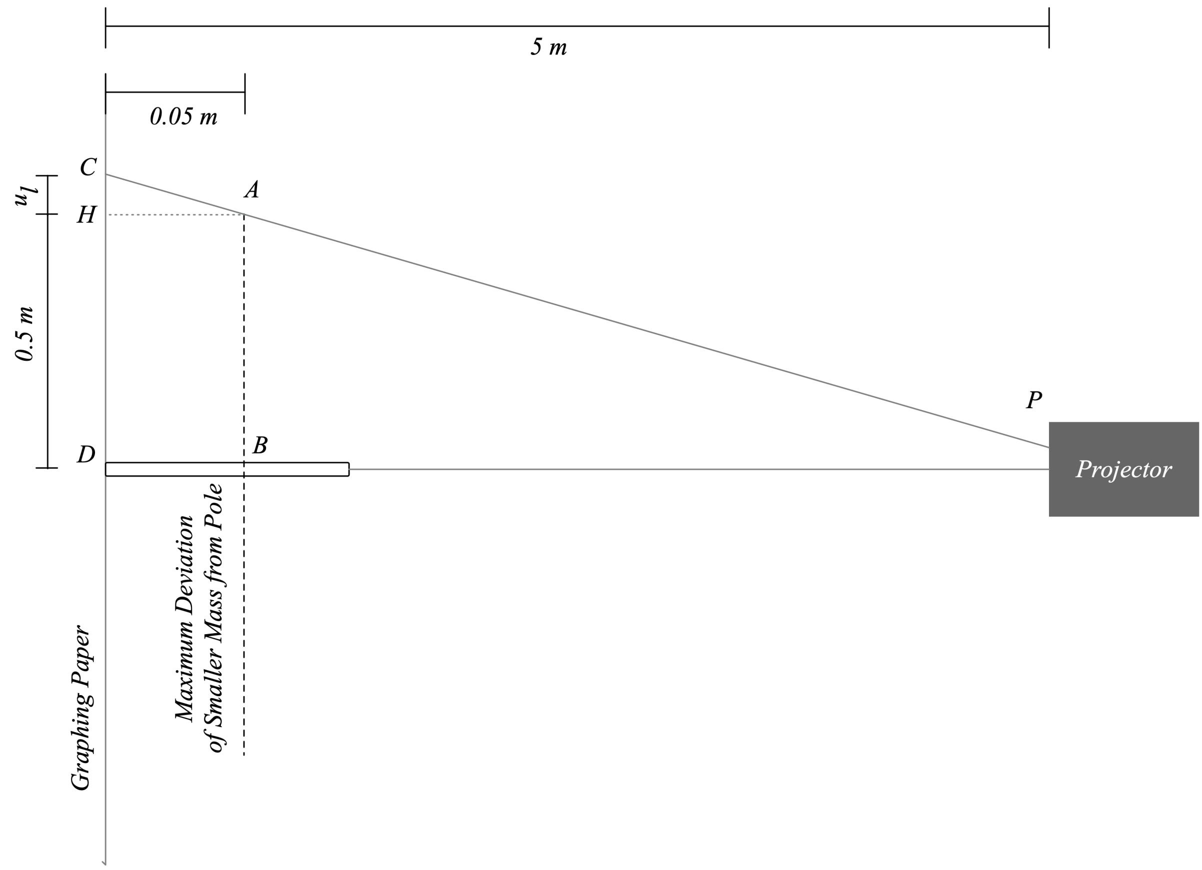
$$t = \frac{\textbf{frame}}{60}$$

and find that its uncertainty is $\pm 0.0005s$ determined from its exposure duration.

*B. Uncertainty Values of x and y*

A Cartesian grid was chosen in order the reduce the uncertainty of the propagated values of $\theta$ and r, as their uncertainties would have been greater if a concentric grid paper was used due to the spacing of angle graticules. The *x* and *y* coordinates were taken from the joint point of the string and the attached mass, with an uncertainty of both ± 1 cm, in order to account for the deviations of the shadow due to the ratio of the distance between the projector to the grid, to the string to the grid. Specifically, due to *Constraint A* and *B* (from Section III-1.), the maximum deviation of the shadow from the mass ($u_l$) can be determined by the similarity of $\triangle PAB$ and $\triangle PCD$ in *Diagram 4.1.1*, as $u_l = 0.005m$.

*[Diagram 4.1.1. Measuring Deviation of Shadow due to Point Light Source Projector, Top View]*



Accounting for minor movements of the pole as well as the thickness of the pole itself, the uncertainty of the *x* and *y* values are determined as ± 0.01 meters.

These values were processed into the polar coordinates that indicate values of $\theta$ and *r*, through the following relationship:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(\frac{y}{x}) + n\pi$$

Where n is determined according to *Table 4.1.1* in order to accommodate for the quadrant of the location of the mass:

*[Table 4.1.1. Quadrant of Location of Mass in Determining Value of n in obtaining θ]*

| Quadrant | I | II | III | IV |
|---|---|---|---|---|
| n | 0 | 1 | 1 | 2 |

The uncertainties of r was calculated by calculating its maximum and minimum values:

$$r_{MAX} = \sqrt{(|x| + \delta x)^2 + (|y| + \delta y)^2}$$
$$r_{MIN} = \sqrt{(|x| - \delta x)^2 + (|y| - \delta y)^2}$$
$$\delta r = \frac{(r_{MAX} + r_{MIN})}{2}$$

Uncertainties of $\theta$ was calculated similarly, first by calculating the uncertainty of $\frac{y}{x}$ which we will denote as $\delta\mathbf{div}$, and then by calculating the minimum and the maximum of $\tan^{-1}(\frac{y}{x} \pm \delta\mathbf{div}) + n\pi$. The choice of y and x from $y_{MAX}$, $y_{MIN}$, $x_{MAX}$, and $y_{MIN}$ are determined by whether they are positive or negative.

Note that despite the fact that the unites of *x* and *y* in *Table 4.2.1* is given in centimeters, for r the final value with uncertainty can be converted into meters (as shown in the graph), and for $\theta$ the units are not a concern as $\frac{y}{x}$ does not have units, and neither does $\theta$.

C. Example Calculation

As an example, for values at $t = 0.1000 \pm 0.0005s$, the values of x and y are respectively $-43 \pm 1cm$ and $-6 \pm 1cm$. The minimum and maximum values of x and y are:

$$y_{MAX} = -42cm \qquad y_{MIN} = -44cm$$
$$x_{MAX} = -7cm \qquad y_{MIN} = -5cm$$

| *(i) Calculating $\theta$* | *(ii) Calculating r* |
|---|---|
| Therefore, in calculating $\delta\textbf{div}$: $$\delta\textbf{div}_{MAX} = \frac{y_{MAX}}{x_{MIN}} = 0.11364$$ $$\delta\textbf{div}_{MIN} = \frac{y_{MIN}}{x_{MAX}} = 0.16667$$ and finally calculate the maximum and minimum of $\theta$ using a suitable value of n: $$\theta_{MAX} = \tan^{-1}(\delta\textbf{div}_{MIN}) + \pi = 3.30673$$ $$\theta_{MIN} = \tan^{-1}(\delta\textbf{div}_{MAX}) + \pi = 3.25474$$ $$\delta\theta = \frac{(\theta_{MAX} + \theta_{MIN})}{2} = 0.026$$ $$\therefore \theta = 3.28 \pm 0.03$$ | Therefore, in calculating r, $$r_{MAX} = \sqrt{(\lvert x \rvert + \delta x)^2 + (\lvert y \rvert + \delta y)^2} = 0.445533m$$ $$r_{MIN} = \sqrt{(\lvert x \rvert - \delta x)^2 + (\lvert y \rvert - \delta y)^2} = 0.422966m$$ And therefore calculate the value or r and its uncertainty: $$\delta r = \frac{(r_{MAX} + r_{MIN})}{2} = 0.0113$$ $$\therefore r = 0.434 \pm 0.011m$$ |

**2. Result Tables**

*[Table 4.2.1. Qualitative Descriptions of Behavior of System at Specific Times]*

| ***Time*** ***t / s*** | ***Quadrant*** | ***Qualitative Description*** |
|---|---|---|
| ±0.0005 | | |
| 0.0167 | III | Immediately after release of string |
| 0.0333 | | Acceleration of smaller and bigger mass due to gravity |
| 0.0500 | | Continued Acceleration of smaller and bigger mass due to gravity |
| 0.0667 ~0.2500 | | *ut supra* |
| 0.2667 | | Reaches lowest observed point in graph. |
| 0.2833 | IV | Friction begins to cause data to noticeably shift (See *Graph 4.3.1*). |
| 0.3000 | | Full rotation of string around pole, mass passes through starting point. |
| 0.3167 | I | Extrapolated graphing paper to find coordinates, by measuring length on video itself. |
| 0.3333 | | Bigger mass is observed to stop moving. |
| 0.3500 | II | Smaller mass continues to wrap around pole. |
| 0.3667 | III | *ut supra* |
| 0.3833 | | End of data collected. Deviations due to friction are significant. |

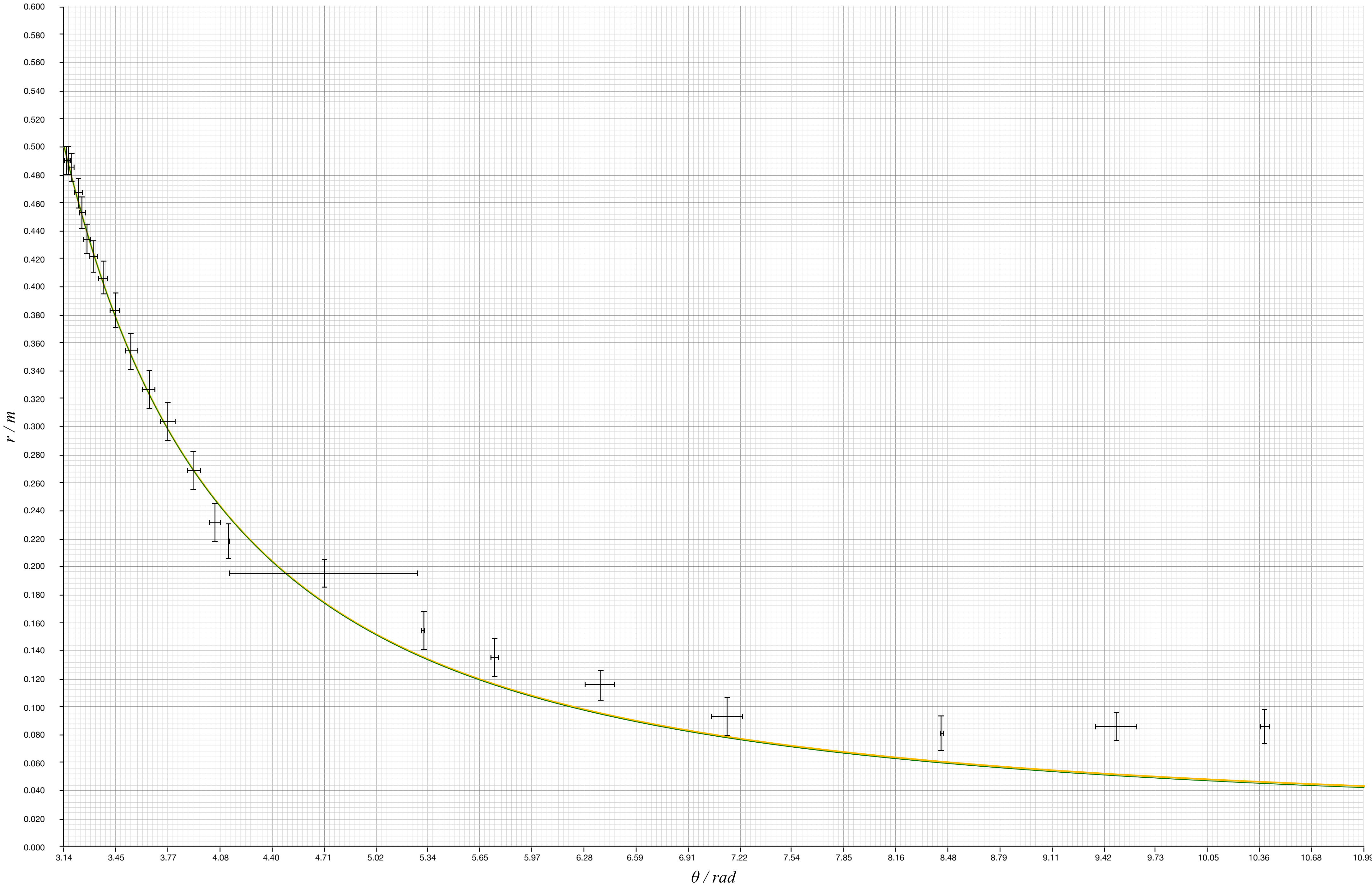
*[Table 4.2.2. Raw and Processed Quantitative Data of Cartesian and Polar Coordinates]*

| Frame Number | Time<br>t / s | x -coordinate<br>x / cm | y-coordinate<br>y / cm | angle between x-axis and string<br>θ / rad | distance from smaller mass to pole<br>r / m |
|---|---|---|---|---|---|
| | ±0.0005 | ±1.0 | ±1.0 | | |
| 1 | 0.0167 | −49.0 | −0.5 | 3.15 ± 0.02 | 0.490 ± 0.010 |
| 2 | 0.0333 | −49.0 | −1.0 | 3.16 ± 0.02 | 0.490 ± 0.010 |
| 3 | 0.0500 | −48.5 | −2.0 | 3.18 ± 0.02 | 0.486 ± 0.010 |
| 4 | 0.0667 | −46.5 | −4.0 | 3.23 ± 0.02 | 0.467 ± 0.011 |
| 5 | 0.0833 | −45.0 | −5.0 | 3.25 ± 0.02 | 0.453 ± 0.011 |
| 6 | 0.1000 | −43.0 | −6.0 | 3.28 ± 0.03 | 0.434 ± 0.011 |
| 7 | 0.1167 | −41.5 | −7.5 | 3.32 ± 0.03 | 0.422 ± 0.012 |
| 8 | 0.1333 | −39.5 | −9.5 | 3.38 ± 0.03 | 0.406 ± 0.012 |
| 9 | 0.1500 | −36.5 | −11.5 | 3.45 ± 0.03 | 0.383 ± 0.013 |
| 10 | 0.1667 | −32.5 | −14.0 | 3.55 ± 0.04 | 0.354 ± 0.013 |
| 11 | 0.1833 | −28.5 | −16.0 | 3.65 ± 0.04 | 0.327 ± 0.014 |
| 12 | 0.2000 | −24.5 | −18.0 | 3.77 ± 0.04 | 0.304 ± 0.014 |
| 13 | 0.2167 | −18.5 | −19.5 | 3.92 ± 0.04 | 0.269 ± 0.014 |
| 14 | 0.2333 | −12.5 | −19.5 | 4.05 ± 0.03 | 0.232 ± 0.014 |
| 15 | 0.2500 | −6.0 | −21.0 | 4.14 ± 0.00 | 0.219 ± 0.012 |
| 16 | 0.2667 | 0.5 | −19.5 | 4.71 ± 0.57 | 0.195 ± 0.010 |
| 17 | 0.2833 | 6.5 | −14.0 | 5.31 ± 0.01 | 0.154 ± 0.013 |
| 18 | 0.3000 | 11.5 | −7.0 | 5.74 ± 0.02 | 0.135 ± 0.014 |
| 19 | 0.3167 | 11.5 | 1.0 | 6.38 ± 0.09 | 0.116 ± 0.011 |
| 20 | 0.3333 | 5.5 | 7.5 | 7.14 ± 0.10 | 0.093 ± 0.014 |
| 21 | 0.3500 | −3.0 | 7.5 | 8.44 ± 0.01 | 0.081 ± 0.013 |
| 22 | 0.3667 | −8.5 | −0.5 | 9.50 ± 0.12 | 0.086 ± 0.011 |
| 23 | 0.3833 | −3.0 | −8.0 | 10.40 ± 0.03 | 0.086 ± 0.013 |

## 3. Graph



*[Graph 4.3.1. Distance Between Smaller Mass and Pole Against Angle Between Reference X-Axis and String Between Smaller Mass and the Pole]*



## V. Conclusion

Comparison of obtained data with the theoretical graph reveals that the experiment has closely matched our theoretical prediction, stated in *Section II*, with the inverse relationship of $r$ and $\theta$ as identified in *Diagram 3.3.1*. We can quantitatively obtain the degree of the data's fit with the theoretical graph with a coefficient of determination:

$$R_{RES} = \Sigma_i[r_t(\theta_i) - r_i]$$
$$= 0.02474$$
$$R_{TOT} = \Sigma_i[r_i - \frac{\Sigma_i r_i}{n}]$$
$$= 4.4647293$$
$$R^2 = 1 - \frac{R_{RES}}{R_{TOT}}$$
$$= 0.9945$$

The high value (0.9945) of the determination coefficient indicates that the empirical data fits the theoretical predictions made through Lagrangian mechanics. We can therefore deduce that our initial theory, stated both descriptively and numerically, is likely—that in this system of an alternate Atwood machine:

*The mass will fall as the string wraps itself around the pole, returning to its original position (having wrapped itself one and a quarter times in relation to the bigger mass) after about 0.3 seconds. The length of the string between the pole and the smaller mass will decrease fast initially, and slow to a value of about 0.07m in this timespan, preventing the fall of the bigger mass and ultimately stopping due to friction.*

The system, however, begins to deviate significantly after the plotted points on the graph ($t > 0.3s$). This is possibly due to the friction that was not accounted for in the Euler-Langrange equations, and therefore the theoretical model predicts that the smaller mass will continue to slip due to the tension imposed by the gravitational pull on the bigger mass. This flaw in our assumption will be evaluated in the following section.

# VI. Evaluation

## 1. Problematic Assumptions

Systematic errors that caused a deviation in the graph was caused by major, but inevitable assumptions in the theory that was not accounted for in the experiment:

1. There is no friction between the string and the tape on the pole
2. The masses are point masses

These assumptions are difficult to take account for in the uncertainties of values. To address them, we can either modify the theory to take them into account, or try to minimize their effect in the experiment as much as possible.

The friction between the string and the pole can be incorporated into the theory by using the *Euler-Eytelwein equation* or the *Capstan equation*, which relates the tensions of a string around a cylinder with the angle swept by the string. However, it is uncertain whether this relationship can be used with Lagrangian mechanics, and may necessitate a reformulation of the theory. There is also a need for a pole and tape smooth enough for the coefficient of friction to be constant, and a method to measure its value.

On the other hand, using a smoother pole, tape, or string will also help to reduce the friction and reduce the observed deviations. However, a quantitative prediction of the reduced uncertainties cannot be determined, as the current theory simply ignores all friction. Therefore, a reformulation taking into account the friction would lead to a more effective prediction descriptive of empirical behavior.

For the second assumption, smaller sized weights with higher densities can be used to reduce the volume of the masses. Weights made out of lead or tungsten may be a better solution, but the former is a safety issue while the latter is fairly expensive to purchase.

The few outlier points in the graph may be caused by these assumptions, as the coefficient of friction may vary along the curve, while the weights may cause an oscillation or bending of the pole that can spontaneously produce outlier points.

## 2. Alternative Methods

Uncertainties accounted for in the experiment are also of concern:

1. x, y: the spatial coordinates had an uncertainty of ±0.01 m, due to the fineness of the graticules as well as to account for pole wobble.
2. t: the time coordinate had an uncertainty of ±0.0005 s, due to the shutter speed of the camera.

The first uncertainty can be reduced by reducing the wobble of the pole using reinforcements; diagonal sticks can be attached to the stands and clamped to the table. We can predict that this can reduce the uncertainty to around ±0.005 m, if suitably executed.

We can also move the projector further away from the apparatus. Using the method shown in Section IV-1 and demonstrated in *Diagram 4.1.1*, if the projector was moved 10m away from the graph paper, the uncertainty would be maximum ±0.003 m. However, this would be ineffective without addressing the wobble.

The second uncertainty can be reduced by using a camera with a faster shutter speed. This was attempted in the experiment, but proved to be difficult as a faster shutter speed results in a darker video. Brighter projector lights can be used to allow for lower shutter speeds, and reduce the exposure duration down to ±0.0004 seconds, with the lowest possible exposure setting on the camera of (1/2400 s). However, this would be meaningless without using a graph paper with finer graticules, as the displacement between each frame would be smaller.

# VII. Bibliography

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# IIX. APPEDIX

## Appendix A: Wolfram Language Code for Obtaining Theoretical Relationship via Solving Euler-Langrange Equations

```
In[101]:= << VariationalMethods`

In[102]:= ClearAll[r];
          ClearAll[θ];
          ClearAll[M];
          ClearAll[m];
          ClearAll[g];
          ClearAll[t];

In[108]:= lagr = 0.5 (M + m) (r'[t])^2 + 0.5 m r[t]^2 (θ'[t])^2 - g r[t] (M + m Sin[θ[t]]);

In[109]:= a = EulerEquations[lagr, θ[t], t]
          b = EulerEquations[lagr, r[t], t]

Out[109]= m r[t] (-g Cos[θ[t]] - 2. r′[t] θ′[t] - 1. r[t] θ″[t]) == 0

Out[110]= -g (M + m Sin[θ[t]]) + 1. m r[t] θ′[t]^2 - 1. (m + M) r″[t] == 0

In[111]:= TraditionalForm[a]
          TraditionalForm[b]
```

Out[111]//TraditionalForm=

$$m\, r(t)\, (-g \cos(\theta(t)) - 2.\, r'(t)\, \theta'(t) - 1.\, r(t)\, \theta''(t)) = 0$$

Out[112]//TraditionalForm=

$$-g\, (m \sin(\theta(t)) + M) - 1.\, (m + M)\, r''(t) + 1.\, m\, r(t)\, \theta'(t)^2 = 0$$

```
In[113]:= DSolve[{a, b}, {θ[t], r[t]}, t];

In[313]:= M = 1.01;
          m = 0.009;
          g = 9.81;
          r0 = 0.5;

          θ0 = π;

          s = NDSolve[{
               r[t] (-g Cos[θ[t]] - 2.` r′[t] θ′[t] - 1.` r[t] θ″[t]) == 0,
               -g (M + m Sin[θ[t]]) + 1.` m r[t] θ′[t]^2 - 1.` (m + M) r″[t] == 0,
               θ[0] == θ0, r[0] == r0, r'[0] == 0, θ'[0] == 0
              }, {θ[t], r[t]}, {t, 0, 10}];

          Plot[θ[t] /. s, {t, 0, 0.4}, PlotRange → All]
          Plot[r[t] /. s, {t, 0, 0.4}, PlotRange → All]
          ParametricPlot[{θ[t] /. s[[1]], r[t] /. s[[1]]},
           {t, 0.3, 0.33}, AspectRatio → 1 / 2, PlotRange → All]
```